Operator inequalities and trace inequalities derived from Tsallis entropies(Recent Developments in Linear Operator Theory and its Applications)

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Tsallis entropy に関する作用素不等式とトレース不等式

Operator inequalities and trace inequalities derived from Tsallis entropies

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1 Trace inequalities of Tsallis entropy

We define q-logarithm function as follows;

$$\ln_q x = \frac{x^{1-q}-1}{1-q}, \quad (x \geq 0, q \geq 0, q \neq 1).$$

Then we have the following properties;

(1) $\lim_{q\to 1} \ln_q x = \log x$. (in uniformly)

(2) $\ln_q xy = \ln_q x + \ln_q y + (1-q)\ln_q x \ln_q y$.

(3) $\ln_q x$: concave in x for $q \geq 0$.

Definition 1 (Tsallis entropy) *For density operator ρ on a finite dimensional Hilbert space $\mathcal{H}$, Tsallis entropy $S_q(\rho)$ is defined by*

$$S_q(\rho) = \frac{Tr[\rho^q - \rho]}{1-q}, \quad (q \geq 0, q \neq 1).$$

Proposition 1 *We have the following properties;*

(1) $\lim_{q\to 1} S_q(\rho) = -Tr[\rho \log \rho]$.

(2) $S_q(\rho_1 \otimes \rho_2) = S_q(\rho_1) + S_q(\rho_2) + (1-q)S_q(\rho_1)S_q(\rho_2)$.

Lemma 1 $S_q(\rho) \le \ln_q d, \quad (d = \dim \mathcal{H})$.

Proof. Since $\ln_q x$ is concave, we have

$$D_q(A|B) = -\sum_{j=1}^{d} a_j \ln_q \frac{b_j}{a_j} \ge -\ln_q\left(\sum_{j=1}^{d} a_j \frac{b_j}{a_j}\right) = 0.$$

We put $A = \{a_j\}$, $B = \{u_j\}$, $u_j = \frac{1}{d}$ $(1 \le j \le d)$. Then

$$D_q(A|B) = -d^{q-1}(S_q(A) - \ln_q d) \ge 0.$$

Thus $S_q(A) \le \ln_q d$. q.e.d.

Lemma 2 *If f is a concave function satisfying $f(0) = f(1) = 0$, then*

$$|f(t+s) - f(t)| \le \max\{f(s), f(1-s)\},$$

where $s \in [0, 1/2], t \in [0,1], 0 \le s+t \le 1$.

Proof. We put

$$r(t) = f(s) - f(t+s) + f(t).$$

Then

$$r'(t) = -f'(t+s) + f'(t).$$

Since f' is a monotone decreasing function, $r'(t) \ge 0$. Thus we have $r(t) \ge 0$ by $r(0) = 0$. Therefore $f(t+s) - f(t) \le f(s)$. We also put

$$\ell(t) = f(t+s) - f(t) + f(1-s).$$

Then

$$\ell'(t) = f'(t+s) - f'(t).$$

Since f' is a monotone decreasing function, $\ell'(t) \le 0$. Thus we have $\ell(t) \ge 0$ by $\ell(1-s) = 0$. Therefore $-f(1-s) \le f(t+s) - f(t)$. Thus we have the result. q.e.d.

Lemma 3 *If* $|u-v| \leq 1/2$, *then* $|\eta_q(u) - \eta_q(v)| \leq \eta_q(|u-v|)$,
where $\eta_q(x) = \dfrac{x^q - x}{1-q}$, $q \in [0,2]$, $u, v \in [0,1]$.

Proof. Since η_q is a concave function with $\eta_q(0) = \eta_q(0)$, we have

$$|\eta_q(t+s) - \eta_q(t)| \leq \max\{\eta_q(s), \eta_q(1-s)\}$$

for $s \in [0, \frac{1}{2}]$ and $t \in [0,1]$ satisfying $0 \leq t+s \leq 1$ by Lemma 2.
Since $\eta_q(x)$ is a monotone increasing function on $[0, q^{1/(1-q)}]$ and $q^{1/(1-q)} \leq \frac{1}{2}$ for $q \in (0,2]$.

$$\max\{\eta_q(s), \eta_q(1-s)\} = \eta_q(s).$$

Thus we have the result by letting $u = t+s$ and $v = t$. q.e.d.

Lemma 4 *Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *be eigenvalues of Hermitian matrix* A *and* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ *be eigenvalues of Hermitian matrix* B. *Then we have the following;*

$$Tr[|A-B|] \geq \sum_{i=1}^{n} |\lambda_i - \mu_i|.$$

Theorem 1 (Generalized Fannes's inequality) *For two density operators* ρ_1, ρ_2 *on* $\mathcal{H}$ *and* $q \in [0,2]$, *if* $\|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)}$, *then*

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1),$$

where $d = \dim \mathcal{H}$ *and* $\|A\|_1 = Tr[|A|]$.

Proof. Let $\lambda_1^{(i)} \geq \cdots \geq \lambda_n^{(i)}$ be eigenvalues of ρ_i.

We set

$$\epsilon = \sum_{j=1}^{d} \epsilon_j, \quad \epsilon_j = |\lambda_j^{(1)} - \lambda_j^{(2)}|.$$

From Lemma 2,

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^{d} |\eta_q(\lambda_j^{(1)}) - \eta_q(\lambda_j^{(2)})| \leq \sum_{j=1}^{d} \eta_q(\epsilon_j).$$

By $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$ and Lemma 1, we have

$$\begin{aligned}\sum_{j=1}^{d}\eta_q(\epsilon_j) &= -\sum_{j=1}^{d}\epsilon_j^q \ln_q \epsilon_j = \epsilon\{-\sum_{j=1}^{d}\frac{\epsilon_j^q}{\epsilon}\ln_q(\frac{\epsilon_j}{\epsilon}\epsilon)\} \\ &= \epsilon\{-\sum_{j=1}^{d}\frac{\epsilon_j^q}{\epsilon}\ln_q\frac{\epsilon_j}{\epsilon} - \sum_{j=1}^{d}\frac{\epsilon_j^q}{\epsilon}(\frac{\epsilon_j}{\epsilon})^{1-q}\ln_q \epsilon\} \\ &= \epsilon_q\sum_{j=1}^{d}\eta_q(\frac{\epsilon_j}{\epsilon}) + \eta_q(\epsilon) \le \epsilon^q \ln_q d + \eta_q(\epsilon).\end{aligned}$$

Therefore we have

$$|S_q(\rho_1) - S_q(\rho_2)| \le \epsilon^q \ln_q d + \eta_q(\epsilon).$$

From Lemma 3, we have $\|\rho_1 - \rho_2\|_1 \ge \epsilon$. And $\eta_q(x)$ is monotone increase on $x \in [0, q^{1/(1-q)}]$. In addition, x^q is monotone increase on $x \in [0, 2]$. Thus we have theorem.

q.e.d.

Since $\lim_{q\to 1} q^{1/(1-q)} = 1/e$, we have

Corollary 1 (Fannes's inequality) *For two density operators ρ_1, ρ_2 on $\mathcal{H}$, if $\|\rho_1 - \rho_2\|_1 \le 1/e$, then*

$$|S_1(\rho_1) - S_1(\rho_2)| \le \|\rho_1 - \rho_2\|_1 \log d + \eta_1(\|\rho_1 - \rho_2\|_1),$$

where $S_1(\rho) = -Tr[\rho\log\rho]$, $\eta_1(x) = -x\log x$.

2 Operator inequalities of Tsallis relative operator entropy

We change the notation ($\lambda = 1 - q$). That is, for $\lambda \in (0, 1]$,

$$\ln_\lambda x = \frac{x^\lambda - 1}{\lambda}.$$

Definition 2 (Tsallis relative operator entropy) *For $A > 0, B > 0, \lambda \in (0, 1]$, Tsallis relative operator entropy $T_\lambda(A|B)$ is defined by*

$$T_\lambda(A|B) = A^{1/2}\ln_\lambda(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Proposition 2 *we have the following properties;*

(1) $\lim_{\lambda\to 0} T_\lambda(A|B) = S(A|B) = A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2}$.

(2) $T_\lambda(\alpha A|\alpha B) = \alpha T_\lambda(A|B),\ \alpha \in \mathbb{R}^+$.

(3) *If* $B \leq C$, *then* $T_\lambda(A|B) \leq T_\lambda(A|C)$.

(4) $T_\lambda(A_1 + A_2|B_1 + B_2) \geq T_\lambda(A_1|B_1) + T_\lambda(A_2|B_2)$.

(5) $T_\lambda(\alpha A_1 + \beta A_2|\alpha B_1 + \beta B_2) \geq \alpha T_\lambda(A_1|B_1) + \beta T_\lambda(A_2|B_2)$.

(6) $T_\lambda(UAU^*|UBU^*) = UT_\lambda(A|B)U^*$.

(7) $\Phi(T_\lambda(A|B)) \leq T_\lambda(\Phi(A)|\Phi(B))$, *where* U *is an unital positive linear map.*

Remark 1 *Same properties are shown for a more general case by Fujii et.al.*

Solodarity $AsB = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$ *for operator monotone* f.

Since
$$\frac{x^{-\lambda}-1}{-\lambda} \leq \log x \leq \frac{x^\lambda - 1}{\lambda}$$
for $x > 0, \lambda > 0$, we have the following.

Proposition 3 *For* $A > 0, B > 0, \lambda \in (0,1]$, *we have the following;*
$$T_{-\lambda}(A|B) \leq S(A|B) \leq T_\lambda(A|B).$$

Since
$$1 - \frac{1}{x} \leq \ln_\lambda x \leq x - 1$$
for $x > 0, 0 < \lambda \leq 1$, we have the following.

Proposition 4 *For* $A > 0, B > 0, \lambda \in (0,1]$, *we have the following;*
$$A - AB^{-1}A \leq T_\lambda(A|B) \leq B - A.$$

Since
$$x^\lambda(1 - \frac{1}{\alpha x}) + \ln_\lambda \frac{1}{\alpha} \leq \ln_\lambda x \leq \frac{x}{\alpha} - 1 - x^\lambda \ln_\lambda \frac{1}{\alpha}$$
for $\alpha > 0, x > 0, 0 < \lambda \leq 1$, we have the following.

Theorem 2 *For $A>0, B>0, \alpha>0, \lambda\in(0,1]$, we have the following;*

$$A\sharp_\lambda B-\frac{1}{\alpha}A\sharp_{\lambda-1}B+(\ln_\lambda\frac{1}{\alpha})A\leq T_\lambda(A|B)\leq\frac{1}{\alpha}B-A-(\ln_\lambda\frac{1}{\alpha})A\sharp_\lambda B,$$

where $A\sharp_\lambda B=A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2}$.

We have the following by taking $\lambda\to 0$, $\alpha=1$, respectively;

Corollary 2 *For $A>0, B>0, \alpha>0$, we have the following;*

$$(1-\log\alpha)A-\frac{1}{\alpha}AB^{-1}A\leq S(A|B)\leq(\log\alpha-1)A+\frac{1}{\alpha}B.$$

For $A>0, B>0$, we have the following;

$$A-AB^{-1}A\leq S(A|B)\leq B-A.$$

Lemma 5 *For $X>0, Y>0, a\in\mathbb{R}$, we have*

$$(X\otimes Y)^a=X^a\otimes Y^a.$$

Theorem 3 *For $A_1, A_2, B_1, B_2>0, \lambda\in(0,1]$, we have the following;*

$$T_\lambda(A_1\otimes A_2|B_1\otimes B_2)=T_\lambda(A_1|B_1)\otimes A_2+A_1\otimes T_\lambda(A_2|B_2)+\lambda T_\lambda(A_1|B_1)\otimes T_\lambda(A_2|B_2).$$

Proof. From Lemma 5, we have for $X>0, Y>0, \lambda\in(0,1]$,

$$\begin{aligned}\ln_\lambda(X\otimes Y) &= (\ln_\lambda X)\otimes I+I\otimes(\ln_\lambda Y)\\ &\quad+\lambda(\ln_\lambda X)\otimes(\ln_\lambda Y).\end{aligned}$$

By putting $X=A_1^{-1/2}B_1A_1^{-1/2}, Y=A_2^{-1/2}B_2A_2^{-1/2}$ and by multiplying $A_1^{1/2}\otimes A_2^{1/2}$ from both sides, we have the theorem. q.e.d.

Corollary 3 *For $A_1, A_2, B_1, B_2>0$, we have*

$$S(A_1\otimes A_2|B_1\otimes B_2)=S(A_1|B_1)\otimes A_2+A_1\otimes S(A_2|B_2).$$

Since we put $B_1 = B_2 = I, A_i = \rho_i$, we have the following;

Corollary 4 (pseudo additivity) *For ρ_1, ρ_2, we have*

$$S_\lambda(\rho_1 \otimes \rho_2) = S_\lambda(\rho_1) + S_\lambda(\rho_2) + \lambda S_\lambda(\rho_1) S_\lambda(\rho_2).$$

Corollary 5 *From Theorem 3 we have the following inequalities;*

(1) *For $\lambda \in (0,1]$ and $0 < A_i \leq B_i$ $(i = 1, 2)$, we have*

(a) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq \lambda T_\lambda(A_1|B_1) \otimes T_\lambda(A_2|B_2)$.

(b) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1|B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2|B_2)$.

(2) *For $\lambda \in (0,1]$ and $0 < B_i \leq A_i$ $(i = 1, 2)$, we have*

(c) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \leq \lambda T_\lambda(A_1|B_1) \otimes T_\lambda(A_2|B_2)$.

(d) $T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1|B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_1|B_2)$.

3 Trace inequalities of Tsallis relative entropy

Definition 3 (Tsallis relative entropy) *For density operators ρ, σ, Tsallis relative entropy is defined by*

$$D_\lambda(\rho|\sigma) = \frac{Tr[\rho - \rho^{1-\lambda}\sigma^\lambda]}{\lambda}, \quad \lambda \in (0,1].$$

Theorem 4 $D_\lambda(\rho|\sigma) \leq -Tr[T_\lambda(\rho|\sigma)]$.

Proof. We remark that

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2} B A^{-1/2})^\alpha A^{1/2}$$

is α power mean. By Theorem 3.4 in Hiai-Petz [3], we have

$$Tr[e^A \sharp_\alpha e^B] \leq Tr[e^{(1-\alpha)A + \alpha B}].$$

for any $\alpha \in [0,1]$. We put $A = \log \rho, B = \log \sigma$.

$$Tr[\rho \sharp_\alpha \sigma] \leq Tr[e^{\log \rho^{1-\alpha} + \log \sigma^\alpha}].$$

We apply Golden-Thompson inequality

$$Tr[e^{A+B}] \leq Tr[e^A e^B]$$

for any Hermitian operators A, B. Then we have

$$Tr[e^{\log \rho^{1-\alpha}+\log \sigma^{\alpha}}] \leq Tr[e^{\log \rho^{1-\alpha}} e^{\log \sigma^{\alpha}}] = Tr[\rho^{1-\alpha}\sigma^{\alpha}].$$

Thus we have

$$Tr[\rho^{1/2}(\rho^{-1/2}\sigma\rho^{-1/2})^{\alpha}\rho^{1/2}] \leq Tr[\rho^{1-\alpha}\sigma^{\alpha}].$$

q.e.d.

Corollary 6 (Hiai-Petz) $Tr[\rho(\log\rho - \log\sigma)] \leq -Tr[\rho\log(\rho^{-1/2}\sigma\rho^{-1/2})]$.

Definition 4 (Tsallis relative entropy) *For positive operators A, B and $0 < \lambda \leq 1$, we define*

$$D_{\lambda}(A\|B) = \frac{Tr[A - A^{1-\lambda}B^{\lambda}]}{\lambda}.$$

Theorem 5 (Generalized Bogoliubov inequality) *For positive operators A, B and $0 < \lambda \leq 1$, we have the following;*

$$D_{\lambda}(A\|B) \geq \frac{Tr[A] - (Tr[A])^{1-\lambda}(Tr[B])^{\lambda}}{\lambda}.$$

Proof. It follows by the application of the Holder's inequality:

$$|Tr[XY]| \leq Tr[|X|^{s}]^{1/s}Tr[|Y|^{t}]^{1/t}$$

for $1 < s, t < \infty$, $1/s + 1/t = 1$. q.e.d.

Corollary 7 (Peierls-Bogoliubov inequality) *For positive operators A, B, we have the following;*

$$Tr[A(\log A - \log B)] \geq Tr[A](\log Tr[A] - \log Tr[B]).$$

References

[1] S.Furuichi, K.Yanagi and K.Kuriyama, Fundamental properties for Tsallis relative entropy, J.Math.Phys, vol.45, pp.4868-4877, 2004.

[2] S.Furuichi, K.Yanagi and K.Kuriyama, A note on operator inequalities of Tsallis relative operator entropy, Linear Algebra Appl., vol.407, pp.19-31, 2005.

[3] F.Hiai and D.Petz, The Golden-Thompson trace inequality is complemented, Linear Algebra Appl., vol.181, pp.153-185, 1993.

[4] K.Yanagi, S.Furuichi and K.Kuriyama, Generalized Shannon inequalities based on Tsallis relative operator entropy, Linear Algebra Appl., vol.394, pp.109-118, 2005.